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A Periodicity Lemma in Linear Diophantine Analysis

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Let $g = (g_1, \ldots, g_r) \geqslant 0$ and $h = (h_1, \ldots, h_r) \geqslant 0$, $g_\rho, h_\rho \in \mathbb{J}$, be two vectors of nonnegative integers and let $\lambda \in \mathbb{J}$, $\lambda \geqslant 0$, $\lambda \equiv 0 \bmod d$, where d denotes g.c.d. $(g_1, \ldots, g_r)$. Define

$$\Lambda(\lambda) = \Lambda(\lambda; g, h) := \min \left\{ \sum_{\rho=1}^{r} x_\rho h_\rho : x_\rho \geqslant 0, x_\rho \in \mathbb{J}, \sum_{\rho=1}^{r} x_\rho g_\rho = \lambda \right\}.$$

It is shown in this paper that $\Lambda(\lambda)$ is periodic in λ with constant jump. If $i \in \{1, \ldots, r\}$ is such that

$$\det \begin{pmatrix} g_i & h_i \\ g_\rho & h_\rho \end{pmatrix} \geqslant 0 \qquad (\rho = 1, \ldots, r),$$

then

$$\Lambda(\lambda + g_i) = \Lambda(\lambda) + h_i$$

holds true for all sufficiently large λ, $\lambda \equiv 0 \bmod d$.

1. INTRODUCTION

Let $r \geqslant 1$ and let $g = (g_1, \ldots, g_r) \geqslant 0$, $h = (h_1, \ldots, h_r) \geqslant 0$ be vectors of nonnegative integers: $g_\rho, h_\rho \in \mathbb{J}$; $g_\rho \geqslant 0$, $h_\rho \geqslant 0$ $(\rho = 1, \ldots, r)$. It is well known (cf. [3]) that the affine hyperplane in $\mathbb{R}^r$

$$g_1 x_1 + \cdots + g_r x_r - \lambda = 0 \tag{1}$$

contains vectors $(x_1, \ldots, x_r)$ with nonnegative integer components x_ρ provided the "parameter" $\lambda \in \mathbb{J}$, $\lambda \geqslant 0$, is sufficiently large and satisfies $\lambda \equiv 0 \bmod d$, where d denotes g.c.d.$(g_1, \ldots, g_r)$.

Consider the linear function

$$f(x_1, \ldots, x_r) = h_1 x_1 + \cdots + h_r x_r .$$

It is in some respects useful (cf. [4]) to have information about the minimum of f on the "nonnegative integer lattice" of the hyperplane (1). Let us denote this minimum for fixed g and h by $\Lambda(\lambda)$.

It is shown in this paper that $\Lambda(\lambda)$ enjoys a "periodicity" property as λ is running through the multiples of d. If $i \in \{1,..., r\}$ is such that

$$g_i h_\rho - g_\rho h_i \geqslant 0$$

for $\rho = 1,..., r$ (such an i exists, as is easily seen), then

$$\Lambda(\lambda + g_i) = \Lambda(\lambda) + h_i \tag{2}$$

holds for all sufficiently large $\lambda \equiv 0 \bmod d$.

Hence a complete insight in the values of $\Lambda(\lambda)$ is possible, once the finitely many quantities $\Lambda(\lambda_0 + \tau d)$ ($\tau = 0, 1,..., (g_i/d) - 1$) are known for some λ_0 large enough. A lower bound N such that (2) holds for $\lambda \geqslant N$ is also given in our main theorem.

2. Preliminaries

Let $r \geqslant 1$ and let $g = (g_1, ..., g_r)$ be a vector with nonnegative integer components; we assume $g_1 \neq 0$, which is no restriction for our results.

Define

$$d_1 = g_1, \quad d_\rho = \text{g.c.d.}(g_1, ..., g_\rho); \quad \hat{d}_{\rho-1} = d_{\rho-1}/d_\rho \qquad (\rho = 2,..., r), \tag{3}$$

$$N_1 = 0, \qquad N_\rho = N_\rho(g_1, ..., g_\rho) = N_{\rho-1} + (\hat{d}_{\rho-1} - 1)(g_\rho - d_\rho) \qquad (\rho = 2,..., r). \tag{4}$$

Note that

$$d_{\rho-1} \equiv 0 \bmod d_\rho, \qquad N_\rho \equiv 0 \bmod d_\rho \qquad (\rho = 2,..., r). \tag{5}$$

LEMMA 1. *Let $\lambda \in \mathbb{J}$ be such that $\lambda \equiv 0 \bmod d_r$ and $\lambda \geqslant N_r$. Then*

(a) *there exists $\kappa \in \mathbb{J}$, $0 \leqslant \kappa \leqslant \hat{d}_{r-1} - 1$, such that*

$$\lambda - \kappa g_r \equiv 0 \bmod d_{r-1} \qquad \textit{and} \qquad \lambda - \kappa g_r \geqslant N_{r-1} \qquad (r \geqslant 2);$$

(b) *λ admits a "nonnegative g-representation" $(a_1, ..., a_r)$,* i.e., *$a_\rho \in \mathbb{J}$ ($\rho = 1,..., r$) satisfy*

$$a_\rho \geqslant 0, \qquad \sum_{\rho=1}^{r} a_\rho g_\rho = \lambda.$$

Proof (cf. [3]). Put $\hat{g}_r = g_r/d_r \in \mathbb{J}$; then the numbers

$$0, \hat{g}_r, 2\hat{g}_r, ..., (\hat{d}_{r-1} - 1)\, \hat{g}_r$$

span all residue classes mod $\hat{d}_{r-1}$ since any relation

$$p\hat{g}_r \equiv q\hat{g}_r \bmod \hat{d}_{r-1}$$

with $p, q \in \mathbb{J}$ satisfying w.l.o.g. $0 \leqslant q < p \leqslant \hat{d}_{r-1} - 1$, would imply $(p - q)\,\hat{g}_r = l\hat{d}_{r-1}$ for some $l \in \mathbb{J}$ and $0 < p - q < \hat{d}_{r-1}$, which is impossible, since $d_r = \text{g.c.d.}(d_{r-1}, g_r)$, i.e., $1 = \text{g.c.d.}(\hat{d}_{r-1}, \hat{g}_r)$.

The integers

$$(\hat{d}_{r-1} - 1)\,\hat{g}_r, (\hat{d}_{r-1} - 1)\,\hat{g}_r - 1, ..., (\hat{d}_{r-1} - 1)\,\hat{g}_r - (\hat{d}_{r-1} - 1) = (\hat{d}_{r-1} - 1)(\hat{g}_r - 1)$$

also span all residue classes mod $\hat{d}_{r-1}$. Therefore for any $\lambda' \geqslant (\hat{d}_{r-1} - 1)(\hat{g}_r - 1)$ there is $\kappa \in \mathbb{J}$, $0 \leqslant \kappa \leqslant \hat{d}_{r-1} - 1$ such that

$$\lambda' - \kappa\hat{g}_r \equiv 0 \bmod \hat{d}_{r-1}, \qquad \lambda' - \kappa\hat{g}_r \geqslant 0. \tag{6}$$

Let $\lambda' = (\lambda - N_{r-1})/d_r$; λ' is an integer in view of (5) and satisfies $\lambda' \geqslant (\hat{d}_{r-1} - 1)(\hat{g}_r - 1)$ by (4). Hence there is κ as in (6), i.e.,

$$\lambda - N_{r-1} - \kappa g_r \equiv 0 \bmod d_{r-1}, \qquad \lambda - N_{r-1} - \kappa g_r \geqslant 0.$$

This proves part (a) of the lemma; (b) follows immediately from (a) by an inductive argument.

LEMMA 2. *Let* $r \geqslant 2$, $g = (g_1, ..., g_r) \geqslant 0$, $g_\rho \in \mathbb{J}$, $g_1 \neq 0$; *let* $\lambda \equiv 0 \bmod d_r$, $\lambda \geqslant N_r$, *and let* $(0, a_2, ..., a_r)$ *be a nonnegative g-representation of* $\lambda + g_1$:

$$\lambda + g_1 = a_2 g_2 + \cdots + a_r g_r. \tag{7}$$

Then there is a nonnegative g-representation $(c_1, ..., c_r)$ *of* λ

$$\lambda = c_1 g_1 + c_2 g_2 + \cdots + c_r g_r \tag{8}$$

such that

$$\sum_{\rho=i+1}^{r} (a_\rho - c_\rho)\, g_\rho \geqslant 0 \qquad (i = 1, ..., r-1). \tag{9}$$

Proof. We proceed by induction on r.

Let $r = 2$, $\lambda + g_1 = a_2 g_2$, and let $\lambda = c_1 g_1 + c_2 g_2$ be any nonnegative g-representation of λ that exists by Lemma 1 since $\lambda \geqslant N_2$. Then

$$(a_2 - c_2)\, g_2 = \lambda + g_1 - \lambda + c_1 g_1 = (1 + c_1)\, g_1 > 0.$$

Assume now that $r \geqslant 3$ and that the lemma is true for all j, $2 \leqslant j \leqslant r - 1$. Let $\kappa = (\kappa_1, ..., \kappa_r)$ be the g-representation of λ obtained by applying Lemma 1 (a) $(r - 1)$-times. Then

$$\lambda - \sum_{\rho=i+1}^{r} \kappa_\rho g_\rho \geqslant N_i \qquad (i = 1, ..., r - 1) \tag{10}$$

holds true by the very construction of κ. If κ satisfies

$$\sum_{\rho=i+1}^{r} (a_\rho - \kappa_\rho)\, g_\rho \geqslant 0 \qquad (i = 1, ..., r - 1), \tag{11}$$

we set $c_\rho = \kappa_\rho$ $(\rho = 1, ..., r)$ and are done. If (11) is false, there is j, $2 \leqslant j \leqslant r - 1$, such that

$$\sum_{\rho=j+1}^{r} (a_\rho - \kappa_\rho)\, g_\rho < 0. \tag{12}$$

(Note that $j = 1$ is impossible since

$$\sum_{\rho=2}^{r} (a_\rho - \kappa_\rho)\, g_\rho = \lambda + g_1 - (\lambda - \kappa_1 g_1) = (1 + \kappa_1)\, g_1 > 0.)$$

Define

$$\tilde{\lambda} = \sum_{\rho=2}^{j} a_\rho g_\rho - g_1 .$$

Then $\tilde{\lambda} \equiv 0 \bmod d_j$; moreover, by (7), (10), and (12),

$$\begin{aligned}\tilde{\lambda} &= \lambda - \sum_{\rho=j+1}^{r} a_\rho g_\rho \qquad\qquad (13)\\ &= \lambda - \sum_{\rho=j+1}^{r} \kappa_\rho g_\rho - \sum_{\rho=j+1}^{r} (a_\rho - \kappa_\rho)\, g_\rho \\ &> N_j .\end{aligned}$$

Since

$$\tilde{\lambda} + g_1 = a_2 g_2 + \cdots + a_j g_j ,$$

we are in a position to apply the induction hypothesis, thus obtaining a nonnegative vector $(\tilde{c}_1, ..., \tilde{c}_j)$ such that

$$\tilde{\lambda} = \sum_{\rho=1}^{j} \tilde{c}_\rho g_\rho , \tag{14}$$

$$\sum_{\rho=i+1}^{j} (a_\rho - \tilde{c}_\rho) g_\rho \geqslant 0 \qquad (i = 1, ..., j-1). \tag{15}$$

Then (13) and (14) provide a representation

$$\lambda = \sum_{\rho=1}^{j} \tilde{c}_\rho g_\rho + \sum_{\rho=j+1}^{r} a_\rho g_\rho .$$

Put

$$\begin{aligned} c_\rho &= \tilde{c}_\rho , \qquad \rho = 1, ..., j, \\ &= a_\rho , \qquad \rho = j+1, ..., r; \end{aligned}$$

then (8) is satisfied, and (9) follows immediately from (15). This completes the proof.

The following simple lemma is stated without proof.

LEMMA 3. *Let $r \geqslant 2$ and let for $g = (g_1, ..., g_r)$, $h = (h_1, ..., h_r)$*

$$D_{\rho,\sigma} = g_\rho h_\sigma - g_\sigma h_\rho = \det \begin{pmatrix} g_\rho & h_\rho \\ g_\sigma & h_\sigma \end{pmatrix} \qquad (\rho, \sigma = 1, ..., r).$$

Then

(a) *there is $i \in \{1, ..., r\}$ satisfying*

$$D_{i,\rho} \geqslant 0 \qquad (\rho = 1, ..., r),$$

(b) *there is an ordering $\{i_1, ..., i_r\}$ of $\{1, ..., r\}$ such that*

$$D_{i_\rho, i_{\rho+1}} \geqslant 0 \qquad (\rho = 1, ..., r-1).$$

3. A PERIODICITY THEOREM

Let $g = (g_1, ..., g_r) \geqslant 0$, $g_\rho \in \mathbb{J}$, and let $\lambda \equiv 0 \bmod d_r$. We introduce a short notation for the set of nonnegative g-representations of λ:

$$\mathfrak{A}(\lambda; g) = \left\{ a = (a_1, ..., a_r) \colon \sum_{\rho=1}^{r} a_\rho g_\rho = \lambda,\ a_\rho \in \mathbb{J},\ a_\rho \geqslant 0,\ \rho = 1, ..., r \right\}.$$

Suppose $\mathfrak{A}(\lambda; g) \neq \varnothing$. Let $h = (h_1, ..., h_r) \geqslant 0$, $h_\rho \in \mathbb{J}$, and define

$$\Lambda(\lambda) = \Lambda(\lambda; g, h) = \min_{a \in \mathfrak{A}(\lambda; g)} \sum_{\rho=1}^{r} a_\rho h_\rho . \tag{16}$$

Let $D_{\rho,\sigma} = g_\rho h_\sigma - g_\sigma h_\rho$ $(\rho, \sigma = 1, ..., r)$ and observe Lemma 3.

THEOREM. *Let $g = (g_1, ..., g_r) \geqslant 0$, $h = (h_1, ..., h_r) \geqslant 0$, be such that*

$$D_{1,\rho} \geqslant 0 \qquad (\rho = 1, ..., r)$$

and

$$D_{\rho,\rho+1} \geqslant 0 \qquad (\rho = 1, ..., r-1). \tag{17}$$

Then, for every $\lambda \geqslant N_r(g_1, ..., g_r) = \sum_{\rho=2}^{r} (d_{\rho-1} - 1)(g_\rho - d_\rho)$, $\lambda \equiv 0 \bmod d_r$, the identity

$$\Lambda(\lambda + g_1) = \Lambda(\lambda) + h_1 \tag{18}$$

holds true.

Proof. It is obvious that the statement of the theorem is not affected if we omit all g_i with $g_i = 0$. Hence we may assume w.l.o.g. $g_\rho > 0$ $(\rho = 1, ..., r)$ in the following proofs. Since the theorem is trivial for $r = 1$, we may also assume $r \geqslant 2$.

(a) Since $\lambda \geqslant N_r$, we have $\mathfrak{A}(\lambda; g) \neq \varnothing$ by Lemma 1. Let

$$\mathfrak{A}_0(\lambda; g) = \{a = (a_1, ..., a_r) \in \mathfrak{A}(\lambda; g) : a_1 = 0\}$$

$$\mathfrak{A}_1(\lambda; g) = \{a = (a_1, ..., a_r) \in \mathfrak{A}(\lambda; g) : a_1 \geqslant 1\}.$$

Obviously

$$\mathfrak{A}(\lambda; g) = \mathfrak{A}_0(\lambda; g) \cup \mathfrak{A}_1(\lambda; g). \tag{19}$$

Suppose $\mathfrak{A}_0(\lambda + g_1; g) \neq \varnothing$ (otherwise the parts (a)–(c) of this proof are dispensable), and let $a \in \mathfrak{A}_0(\lambda + g_1; g)$, i.e.,

$$\lambda + g_1 = \sum_{\rho=2}^{r} a_\rho g_\rho . \tag{20}$$

We shall now prove that

$$\sum_{\rho=2}^{r} a_\rho h_\rho \geqslant \Lambda(\lambda) + h_1 . \tag{21}$$

To this end let $c = (c_1, \dots, c_r) \in \mathfrak{A}(\lambda; g)$ be such that

$$\sum_{\rho=i+1}^{r} (a_\rho - c_\rho)\, g_\rho \geqslant 0 \qquad (i = 1, \dots, r-1) \tag{22}$$

in accordance with Lemma 2.

(b) We claim

$$D_{1,i} \sum_{\rho=i}^{r} (a_\rho - c_\rho)\, g_\rho \geqslant g_i \sum_{\rho=1}^{i-1} (c_\rho - a_\rho)\, D_{1,\rho} \quad (i = 2, \dots, r). \tag{23}$$

Before proving this we note that

$$g_i D_{j,k} + g_j D_{k,i} + g_k D_{i,j} = 0, \qquad (i, j, k \in \{1, \dots, r\});$$

in particular,

$$g_i D_{1,i+1} + g_1 D_{i+1,i} + g_{i+1} D_{i,1} = 0 \qquad (i = 1, \dots, r-1),$$

which implies

$$g_i D_{1,i+1} \geqslant g_{i+1} D_{1,i} \qquad (i = 1, \dots, r-1) \tag{24}$$

in view of (17).

Inequality (23) is shown by induction on i. If $i = 2$, we have

$$\begin{aligned} D_{1,2} \sum_{\rho=2}^{r} (a_\rho - c_\rho)\, g_\rho &= D_{1,2}(\lambda + g_1 - (\lambda - c_1 g_1)) \\ &= D_{1,2}(1 + c_1)\, g_1 \geqslant 0 = g_2(c_1 - a_1)\, D_{1,1}. \end{aligned}$$

Suppose now that $i \leqslant r - 1$ and that (23) holds. Then, by (22), (24), and the induction hypothesis,

$$\begin{aligned} &D_{1,i+1} \sum_{\rho=i+1}^{r} (a_\rho - c_\rho)\, g_\rho \\ &\geqslant (g_{i+1}/g_i)\, D_{1,i} \sum_{\rho=i+1}^{r} (a_\rho - c_\rho)\, g_\rho \\ &= (g_{i+1}/g_i)\, D_{1,i} \sum_{\rho=i}^{r} (a_\rho - c_\rho)\, g_\rho - g_{i+1} D_{1,i}(a_i - c_i) \\ &\geqslant (g_{i+1}/g_i)\, g_i \sum_{\rho=1}^{i-1} (c_\rho - a_\rho)\, D_{1,\rho} - g_{i+1} D_{1,i}(a_i - c_i) \\ &= g_{i+1} \sum_{\rho=1}^{i} (c_\rho - a_\rho)\, D_{1,\rho}. \end{aligned}$$

Note that (23) yields for $i = r$ (recall that $D_{1,1} = 0$)

$$\sum_{\rho=2}^{r} (a_\rho - c_\rho) D_{1,\rho} \geqslant 0. \tag{25}$$

(c) Observing that $\sum_{\rho=1}^{r} c_\rho g_\rho = \lambda$, we have

$$\sum_{\rho=1}^{r} c_\rho h_\rho = c_1 h_1 + \sum_{\rho=2}^{r} c_\rho h_\rho = (1/g_1)\left(\lambda - \sum_{\rho=2}^{r} c_\rho g_\rho\right) h_1 + \sum_{\rho=2}^{r} c_\rho h_\rho$$

$$= (1/g_1)\left(\lambda h_1 + \sum_{\rho=2}^{r} c_\rho D_{1,\rho}\right).$$

Similarily, by (20),

$$\sum_{\rho=2}^{r} a_\rho h_\rho = (1/g_1)\left((\lambda + g_1) h_1 + \sum_{\rho=2}^{r} a_\rho D_{1,\rho}\right).$$

Hence, by (25),

$$\sum_{\rho=2}^{r} a_\rho h_\rho - \sum_{\rho=1}^{r} c_\rho h_\rho = (1/g_1)\left(g_1 h_1 + \sum_{\rho=2}^{r} (a_\rho - c_\rho) D_{1,\rho}\right) \geqslant h_1 ,$$

which provides (21):

$$\sum_{\rho=2}^{r} a_\rho h_\rho \geqslant \sum_{\rho=1}^{r} c_\rho h_\rho + h_1 \geqslant \Lambda(\lambda) + h_1 .$$

(d) Now let $a = (a_1, ..., a_r) \in \mathfrak{A}_1(\lambda + g_1 ; g)$; then $(a_1 - 1, a_2, ..., a_r) \in \mathfrak{A}(\lambda; g)$, and we obtain at once

$$\sum_{\rho=1}^{r} a_\rho h_\rho = (a_1 - 1) h_1 + \sum_{\rho=2}^{r} a_\rho h_\rho + h_1 \geqslant \Lambda(\lambda) + h_1 . \tag{26}$$

(e) In view of (19) the inequalities (21) and (26) show that

$$\Lambda(\lambda + g_1) \geqslant \Lambda(\lambda) + h_1 .$$

On the other hand, let $b = (b_1, ..., b_r) \in \mathfrak{A}(\lambda; g)$ represent $\Lambda(\lambda)$, i.e., $\sum_{\rho=1}^{r} b_\rho h_\rho = \Lambda(\lambda)$. Then obviously $(b_1 + 1, b_2, ..., b_r) \in \mathfrak{A}(\lambda + g_1 ; g)$, which means

$$\Lambda(\lambda + g_1) \leqslant (b_1 + 1) h_1 + \sum_{\rho=2}^{r} b_\rho h_\rho = \Lambda(\lambda) + h_1 .$$

This completes the proof.

Next we prove a lemma in order to add a generalizing corollary to the theorem.

LEMMA 4. *Let $p > 0$, $p \equiv 0 \bmod d_r$ be minimal in the sense that there exist integers $N \geqslant 0$ and $q \geqslant 0$ satisfying*

$$\Lambda(\lambda + p) = \Lambda(\lambda) + q \tag{27}$$

for every $\lambda \equiv 0 \bmod d_r$, $\lambda \geqslant N$. If $p' > 0$, $p' \equiv 0 \bmod d_r$ is such that

$$\Lambda(\lambda + p') = \Lambda(\lambda) + q' \tag{28}$$

holds true for some $q' \geqslant 0$ and every $\lambda \equiv 0 \bmod d_r$, $\lambda \geqslant N'$, then there is $k \in \mathbb{J}$ such that

$$p' = kp, \qquad q' = kq.$$

Proof. Let $k > 0$ and $\hat{p}$, $0 \leqslant \hat{p} < p$, be such that $p' = kp + \hat{p}$, where $\hat{p} \equiv 0 \bmod d_r$. Let $\lambda \geqslant \hat{N} = \max(N - \hat{p}, N')$, $\lambda \equiv 0 \bmod d_r$. Then

$$\Lambda(\lambda + p') = \Lambda(\lambda + \hat{p} + kp) = \Lambda(\lambda + \hat{p}) + kq,$$

which, in view of (28), means that

$$\Lambda(\lambda + \hat{p}) = \Lambda(\lambda) + q' - kq, \tag{29}$$

where $\hat{q} = q' - kq \geqslant 0$, as is easily seen. Equation (29) contradicts the minimality property of p except for $\hat{p} = 0$, which implies $\hat{q} = 0$.

COROLLARY. *Let g and h be as in the Theorem and assume*

$$D_{1,\rho} = 0 \qquad (\rho = 1,..., j) \tag{30}$$

for some j, $1 \leqslant j \leqslant r$. Then

$$\Lambda(\lambda + d_j) = \Lambda(\lambda) + (h_1/g_1)\, d_j$$

holds for sufficiently large λ, $\lambda \equiv 0 \bmod d_r$.

Proof. The corollary follows immediately from the theorem and from Lemma 4, since (30) implies

$$D_{i,\rho} \geqslant 0 \qquad (\rho = 1,..., r)$$

for every $i \in \{1,..., j\}$.

We remark that $(h_1/g_1)\, d_j \in \mathbb{J}$. We have, by (30),

$$\frac{g_1}{d_j} h_\rho = \frac{g_\rho}{d_j} h_1, \qquad \text{i.e.,} \qquad \frac{g_1}{d_j} \,\Big|\, \frac{g_\rho}{d_j} h_1 \qquad (\rho = 1,..., j);$$

hence g_1/d_j is a divisor of g.c.d.$(g_1h_1/d_j ,..., g_jh_1/d_j) = h_1$.

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